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Radiation in boundary layer flow of an absorbing, emitting and anisotropically scattering fluid

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Abstract *Radiative heat transfer in the laminar boundary layer flow of an absorbing, emitting and anisotropically scattering gray fluid over a flat plate, with the surface of the plate reflecting radiation in diffuse-cum-specular fashion is analyzed. The discrete ordinates method is used to model the radiative transfer. The governing dimensionless momentum and energy equations, in the form of a partial differential system, are solved by a finite difference method. The effect of various parameters like, emittance, the degree of anisotropy in scattering, scattering albedo and the nature of surface reflection on the total heat flux from the plate to the fluid are studied and results are presented.*

Nomenclature

c_p	= specific heat of the fluid at constant pressure, J/kg–K
f	= dimensionless stream function as defined by equation (7)
g	= acceleration due to gravity, m/sec^2, dimensionless intensity function
$I_b(T)$	= black body radiation intensity, W/m^2–sr
$I^+(\tau, \mu, \xi)$	= radiation intensity for positive values of μ, W/m^2–sr
$I^-(\tau, -\mu, \xi)$	= radiation intensity for negative values of μ, W/m^2–sr
k	= thermal conductivity, W/m–K
L	= characteristic length, m
N_c	= conduction-radiation number = $k\beta/4\sigma T_\infty^3$
$p(\mu, \mu')$	= slab scattering phase function
$p(\mu_p)$	= single-scattering phase function
Pr	= Prandtl number = ν/α
q	= heat flux, W/m^2
Re	= Reynolds number, $u_\infty x/\nu$
Q	= dimensionless heat flux = $q/4\sigma T_\infty^4$
T	= temperature, K
T_w	= flat plate temperature, K
T_∞	= fluid free stream temperature, K
u, v	= x and y direction velocity components respectively, m/s
x, y	= coordinates respectively in the parallel and perpendicular directions of the plate, m

Greek symbols

α	= thermal diffusivity = $k/\rho c_p$, m^2/s
β	= extinction coefficient = $\kappa + \gamma$, m^{-1}
ε	= emittance of plate surface
γ	= scattering coefficient, m^{-1}
κ	= absorption coefficient, m^{-1}
μ	= $cos\phi$
ν	= kinematic viscosity, m^2/s
ω	= scattering albedo = γ/β

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ρ = density, kg/m^3
ρ^d = diffuse reflectance of plate surface = $1 - \varepsilon - \rho^s$
ρ^s = diffuse reflectance of plate surface = $1 - \varepsilon - \rho^d$
σ = Stefan-Boltzmann constant, W/m^2–K^4
τ = optical depth of the medium at y defined by $d\tau = \beta dy$
ϕ = polar angle, angle between the radiation vector and the y-axis
θ = dimensionless temperature = T/T_∞
θ_w = T_w/T_∞
η = dimensionless distance as defined by equation (5)
ξ = dimensionless axial distance as defined by equation (6)

Introduction

Radiative heat transfer plays an important role in many engineering applications involving external flows like, atmospheric re-entry, ablative cooling, metalized solid rocket, shock waves, etc. The interaction of radiation with laminar forced convection from a flat plate was modeled by Taitel and Hartnett (1966) for an absorbing and emitting fluid employing exact formulation for the radiative transfer. The analysis including isotropic scattering has been described in the book by Özisik (1973) in which the radiation part is solved exactly using the normal mode expansion technique. Yücel *et al.* (1989) used the P_3 spherical harmonics method to solve the radiation part of the problem while investigating the boundary layer flow of a non-gray, absorbing and emitting radiating fluid. Most of the work on the interaction of radiation with convection in external flow fields is concerned either with absorbing and emitting medium or at most isotropic scattering medium in the case where the scattering effect was considered. The analysis discussing anisotropic scattering and reflecting plate surface has not been reported earlier. The complicated partial differential and integro-differential systems defining the governing equations are extremely difficult to solve, they prompted various approximations in the past to make the problem amenable to a solution (Özisik, 1973). The objective of the present paper is to investigate the effects of the degree of anisotropy in radiation scattering and the nature of reflection (fully diffuse, fully specular, or diffuse-cum-specular) of radiation from the plate surface, on the heat flux when combined laminar forced convection and radiation takes place from a heated flat plate to an absorbing, emitting and anisotropically scattering fluid. The discrete ordinates method is used to solve accurately the radiation part of the problem. The partial differential system defining the governing dimensionless momentum and energy equations is solved by a finite difference method.

Mathematical model

Figure 1 illustrates the flow configuration. A radiatively absorbing, emitting and scattering gray fluid at a free stream temperature T_∞ and velocity u_∞ flows over an isothermal gray flat plate at temperature T_w. Heat is transferred from the plate to the fluid by a combined mode of forced convection and radiation. The net conduction heat transfer in the x-direction is neglected assuming the Peclet number $u_\infty L/\alpha >> 1$. It is also assumed that there is no radiative flux in the x-direction by incorporating the condition of large radiation Peclet number, i.e. $3\beta\rho c_p u_\infty L/16\sigma T^3 >> 1$ (Sparrow and Cess, 1978). The governing equations for a steady, two-dimensional, laminar boundary layer flow are as follows (Özisik, 1973):

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_p}\frac{\partial q^r}{\partial y} \tag{3}$$

Equations (1)-(3) are subject to the boundary conditions:

$$u=v=0 \qquad at \qquad y=0 \tag{4a}$$

$$u=u_\infty \quad at \qquad y\rightarrow\infty \tag{4b}$$

$$T=T_w \qquad at \qquad y=0 \tag{4c}$$

$$T=T_\infty \qquad at \quad x=0 \; and \quad y\rightarrow\infty \tag{4d}$$

The above system of partial differential equations are now transformed into a new system of equations by introducing the following variables:

$$\eta=y\sqrt{\frac{u_\infty}{\nu x}} \tag{5}$$

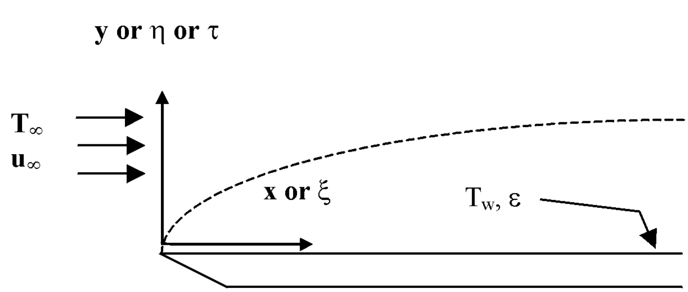

Figure 1.
Flow configuration

$$\xi = \frac{4\sigma T_\infty^3 \beta x}{\rho c_p u_\infty} \tag{6}$$

$$f(x,y) = \frac{\psi(x,y)}{\sqrt{\nu u_\infty x}} \tag{7}$$

ξ is the dimensionless axial distance, characterizing the relative importance of radiation over convection. ψ is the stream function whose partial derivatives yield the x and y velocity components as

$$u = \frac{\partial\psi(x,y)}{\partial y}, \qquad v = -\frac{\partial\psi(x,y)}{\partial x} \tag{8}$$

Substitution of the above variables in equation (1)-(3) leads to the following transformed equations.

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \tag{9}$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2} f \frac{\partial f}{\partial\theta} = \frac{df}{d\eta}\xi\frac{\partial\theta}{\partial\xi} + \xi\frac{\partial Q^r}{\partial\tau} \tag{10}$$

The transformed boundary conditions derived from equation (4) are:

$$f = \frac{\partial f}{\partial\eta} = 0 \quad at \quad \eta = 0 \tag{11a}$$

$$\frac{\partial f}{\partial\eta} = 1 \quad at \quad \eta \to \infty \tag{11b}$$

$$\theta = 1 \quad at \quad \eta = 0 \tag{11c}$$

$$\theta = \theta_\infty \quad at \quad \eta \to \infty \tag{11d}$$

$$\theta = \theta_0 \quad at \quad \xi = 0 \tag{11e}$$

where θ_0 is the solution to the pure convection problem when there is no radiation.

Radiative flux calculation

The radiative transfer equation describing the intensity variation for an absorbing, emitting, scattering, gray, semi-infinite medium, plane-parallel (perpendicular to the y-axis) may be written as (Özisik, 1973)

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \xi) + I(\tau, \mu, \xi) = (1-\omega) I_b[T(\tau, \xi)] + \frac{\omega}{2} \int_{-1}^{1} p(\mu, \mu') I(\tau, \mu', \xi) d\mu' \quad (12)$$

The boundary conditions are

$$I(0, \mu, \xi) = \varepsilon I_b(T_w) + 2\rho^d \int_0^1 I(0, -\mu, \xi) \mu d\mu + \rho^s I(0, -\mu, \xi), \quad \mu > 0 \quad (13)$$

$$I(\tau = \infty, -\mu, \xi) = I(\tau = \infty, \mu, \xi) \quad (14)$$

The optical depth variable τ is defined as $d\tau = \beta d\eta$. The relation between τ, η and ξ is given as (Özisik, 1973)

$$\tau = \eta \sqrt{\xi N_c \mathrm{Pr}} \quad (15)$$

Equation (14) means that at τ(or η) = ∞, the radiative flux vanishes. For numerical computations ∞ is represented by $\tau_{max} = \eta_{max}(\xi N_c \mathrm{Pr})^{1/2}$, where η_{max} is a sufficiently large value representing the edge of the boundary layer. In the subsequent paragraphs, the variable ξ is omitted while showing the variable $I(\tau, \mu, \xi)$, for brevity sake, because we are only concerned with the evaluation of intensity at a particular ξ station.

The discrete ordinates method (DOM) (Modest, 1993) is used to transform the integro-differential equation (12) into a system of coupled linear ordinary differential equations (ODEs). We follow the implementation of DOM employed by Love and Grosh (1965). For convenience, $I(\tau, \mu)$ is separated into a forward component $I(\tau, \mu)$, $\mu \in (0, 1)$ and a backward component $I(\tau, -\mu)$, $\mu \in (0, 1)$ and an m-point Gauss-Legendre numerical quadrature rule is applied to evaluate the scattering integral in equation (12) for each component. This produces a system of $2m$ ODEs in the form:

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} + I(\tau, \mu_i) = (1-\omega) I_b[T(\tau)] + \frac{\omega}{2} \sum_{j=1}^{m} w_j [p(\mu_i, \mu_j) I(\tau, \mu_j) + p(\mu_i, -\mu_j) I(\tau, -\mu_j)] \quad (16)$$

$$-\mu_i \frac{dI(\tau, -\mu_i)}{d\tau} + I(\tau, -\mu_i) = (1-\omega) I_b[T(\tau)] + \frac{\omega}{2} \sum_{j=1}^{m} w_j [p(\mu_i, -\mu_j) I(\tau, \mu_j) + p(\mu_i, \mu_j) I(\tau, -\mu_j)] \quad (17)$$

where w_i, μ_i, $i = 1, m$ are the weights and abscissas of the quadrature rule.

The quadrature rule is applied to the boundary conditions equations (13)-(14) also

$$I(0,\mu_i) = \varepsilon I_b(T_w) + 2\rho^d \sum_{j=1}^{m} w_j \mu_j I(0,-\mu_j) + \rho^s I(0,-\mu_i) \tag{18}$$

$$I(\tau_{\max},-\mu_i) = I(\tau_{\max},\mu_i) \tag{19}$$

Equations (16)-(19) constitute a boundary value problem of ODEs with $2m$ unknowns. We convert this into two initial value problems for ODEs of size m each, one for the forward component $\{I(0,\ \mu_i),\ i = 1,m\}$ and the other for the backward component $\{I(\tau, -\mu_i),\ i = 1,m\}$. First a distribution for $I(\tau, -\mu_i)$ at a number of equidistant points in the interval $(0 - \tau_{max})$ is assumed and the initial values $I(0,\ \mu_i)$ are calculated from equation (18) and the resulting initial value problem for $I(\tau, \mu_i)$ (defined by equations (16) and (18)) is integrated forward from $\tau = 0$ to $\tau = \tau_{max}$. After obtaining $I(\tau_{max}, \mu_i)$ the initial values for $I(\tau_{max}, -\mu_i)$ are evaluated from equation (19) and the initial value problem defined by equations (17) and (19) for $I(\tau, -\mu_i)$ is integrated backward from $\tau = \tau_{max}$ to $\tau = 0$. The procedure is repeated until convergence is achieved for $I(\tau, -\mu_i)$ and $I(\tau, -\mu_i)$ at all τ points considered. We use a fixed step size Crank-Nicolson (CN) semi-implicit scheme for the numerical solution of the ODE. Higher order implicit rules would give better accuracy, however for the present problem the simple and stable CN scheme has been found to be sufficient. The CN method for the present problem needs the inversion of two m-order matrices, but this needs to be done only once in the beginning of the procedure and can be reused later in all steps and iterations. The iterative scheme described earlier to solve the discrete ordinates equations is tailored with the Ng-acceleration scheme described by Auer (1987) to speed up convergence.

Equation (10) contains the divergence of the radiative flux as the last term. This may be evaluated as (Modest, 1993)

$$\frac{\partial Q^r}{\partial \tau} = (1-\omega)\left[\theta^4(\eta) - g(\eta)\right] \tag{20}$$

Where $g(\eta)$ is the dimensionless incident radiation function evaluated from the radiation intensity distribution $I(\tau(\eta, \mu)$ as (Modest, 1993):

$$g(\eta) = \frac{\pi}{2\sigma T_\infty^4}\int_{-1}^{1} I(\tau(\eta),\mu)d\mu = \frac{\pi}{2\sigma T_\infty^4}\sum_{j=1}^{m} w_j[I(\tau(\eta),\mu_j) + I(\tau(\eta),-\mu_j)] \tag{21}$$

The dimensionless radiative heat flux Q^r may now be obtained as:

$$Q^r = \frac{q^r}{4\sigma T_\infty^4} = \frac{\pi}{2\sigma T_\infty^4}\int_{-1}^{1} I(\tau,\mu)\mu d\mu = \frac{\pi}{2\sigma T_w^4}\sum_{j=1}^{m} w_j\mu_j[I(\tau,\mu_j) - I(\tau,-\mu_j)] \tag{22}$$

The total heat flux at the wall is the sum of the convective and radiative contributions and is given as

$$q_w^t = \left[-k\frac{\partial T}{\partial y} + q^r\right]_{y=0} \tag{23}$$

A local Nusselt number may now be defined to represent the total heat flux from the wall to the fluid as:

$$Nu = \frac{q_w^t x}{k(T_w - T_\infty)} \tag{24}$$

Substitution of equations (23) in (24) results in:

$$\frac{Nu}{\sqrt{Re}} = \frac{1}{\theta_w - 1}\left[-\frac{\partial\theta}{\partial\eta} + \sqrt{\frac{\xi\,\mathrm{Pr}}{N_c}}Q^r\right]_{\eta=0} \tag{25}$$

Numerical scheme

Solution of the differential system equations (9) and (10) along with the boundary conditions equation (11) obtains the velocity and temperature profile within the boundary layer. Since the flow field is independent of the temperature, the dimensionless momentum equation, equation (9) is decoupled from the energy equation, equation (10). The solution of equation (9) is, however, available in standard textbooks (Kays and Crawford, 1993). Finite differencing in the ξ variable converts equation (10) into a two-point boundary value problem in ordinary differential equations (ODEs) in the independent variable η. Simple finite differencing is used, i.e. $\left.\frac{\partial\theta}{\partial\xi}\right|_k = \frac{\theta_k - \theta_{k-1}}{\Delta\zeta}$, etc. (Özisik, 1994)

This makes the ODEs at every ξ_k station involve the θ values at the previous ξ station. The resulting two-point boundary value problem is solved using a finite difference method by replacing the derivatives with respect to the variable η with a second order difference scheme (Özisik, 1994). This leads to a non-linear algebraic system, with the nodal temperatures as the unknowns, whose solution was found using the Newton-Raphson method. The value η_{max} = 20 was found to be sufficient for the θ profile to approach θ_∞ asymptotically. Forty equispaced grid points along the η variable were used for calculations. Along the ξ direction a nonuniform grid spacing starting with $\Delta\xi$ = 0.01 increasing in a geometrical progression by a factor of 1.1 at every succeeding grid was used. These grid parameters were found to give satisfactory solution such that no further grid refinement was actually called for. The converged θ solution from the previous ξ station was used as the initial guess to the current ξ station and the procedure was iterated until convergence, i.e. the maximum norm of the vector of relative temperature differences between two succeeding iterations is within 1×10^{-4}.

Results and discussion

The results for black plate and unscattering medium, calculated using the exact formulation for $\partial Q^r/\partial\tau$, is illustrated in Figure 2 together with the DOM results. Both the results are in excellent agreement and also they compare well with the exact results of Zamuraev (1964) as presented by Modest (1993, Figure 20-14, p. 736), thus rendering validation of the present numerical algorithm. Also for ξ = 0, i.e. for pure convection problem, the value of $Nu/Re^{1/2}$ calculated was 0.332, which tallies exactly with that quoted in literature (Kays and Crawford, 1993). The DOM algorithm for the radiation analysis has been validated by comparing the results of the present study, for the radiative flux between two opposing infinite black parallel plates under radiative equilibrium, against the exact results presented by Özisik (1973). The deviation in results is less than 0.1 percent, lending support to the accuracy of the radiative transfer analysis.

To investigate the effect of anisotropic radiation scattering in the total heat transfer we consider a linear anisotropic scattering (LAS) model. Including any kind of scattering model is however quite straightforward. The single-scattering phase function for the LAS model is given as:

$$p(\mu_p) = 1 + a_1\mu_p$$

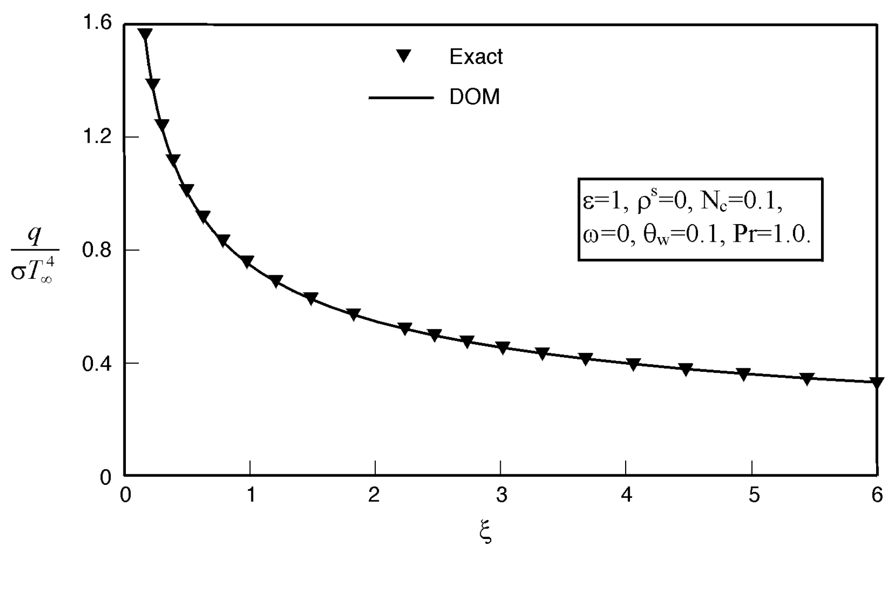

Figure 2.
Comparison of heat flux results

where μ_p is cosine of the angle between the in-scattering and out-scattering directions. $a_1 = 0$ represents isotropic scattering, $a_1 = +1$ indicates strong forward scattering, whereas $a_1 = -1$ corresponds to strong backward scattering. Forward scattering enhances the radiative transfer in the direction from the plate to fluid, while backward scattering retards the transfer. The slab phase function $p(\mu, \mu')$ required in equations (16)-(17) can be easily obtained from $p(\mu_p)$ (Modest, 1993).

The variation of $Nu/Re^{1/2}$ with the dimensionless axial distance ξ for various phase functions and different scattering albedo values is shown in Figure 3. $Nu/Re^{1/2}$ is seen to increase with ξ because of the pronounced radiation effects at higher ξ. The effect of degree of anisotropy a_1 is also demonstrated in Figure 3. For $a_1 = +1$, i.e. for strong forward scattering the increase in $Nu/Re^{1/2}$ over isotropic scattering is insignificant even at the low value of N_c, where radiation dominates over conduction. Strong backscattering ($a_1 = -1$) also affects only insignificant reduction in $Nu/Re^{1/2}$. Increasing scattering albedo decreases the heat flux as observed from Figure 3. As ω increases the heat flux tends to be that of non-radiating flow in view of the increased decoupling between the convective and radiative fluxes. The condition $\omega = 1$ yields the least (and the same) heat flux for all scattering phase functions, in view of the total decoupling of convective and radiative heat fluxes for this case. Specular or diffuse reflection of radiation from the plate surface, even at higher reflection rates, appears to have negligible impact on the heat flux as indicated by the charts presented in Figure 4. The effect of emittance on heat flux is depicted in Figure 5.

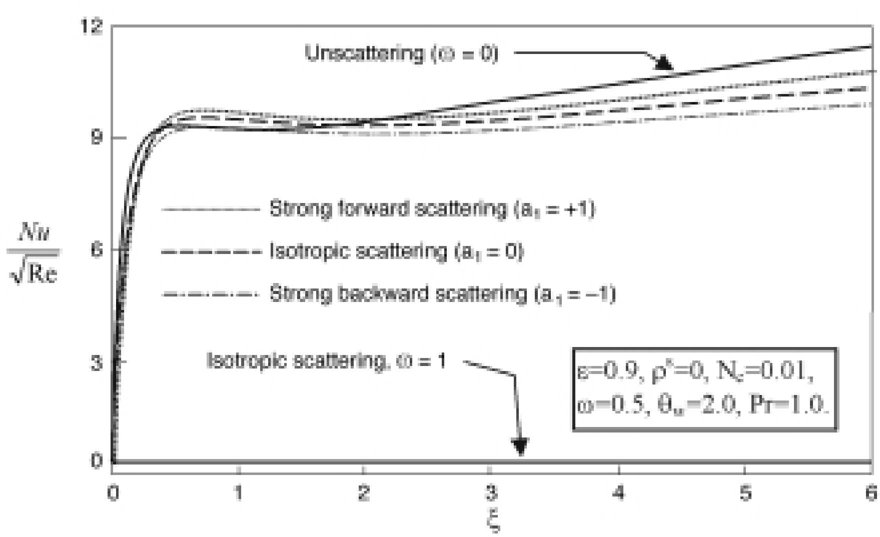

Figure 3.
Effect of scattering phase functions and albedo

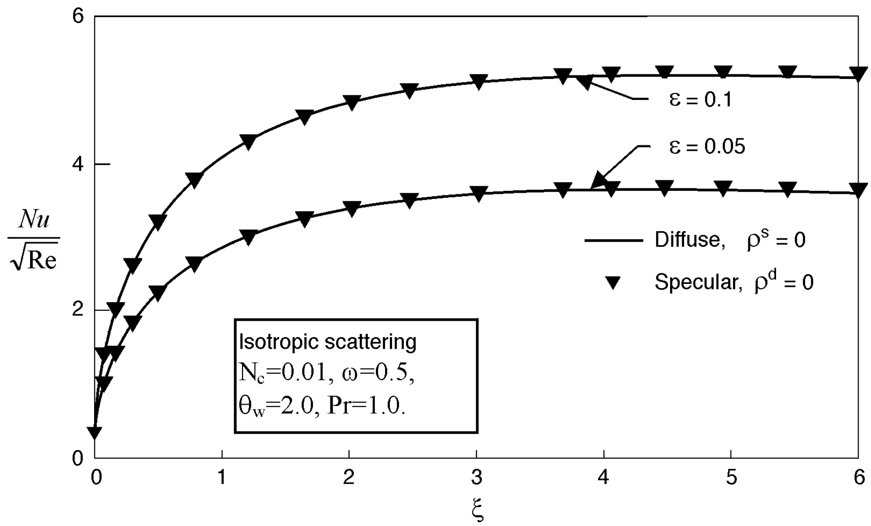

Figure 4.
Effect of nature of reflection from the plate

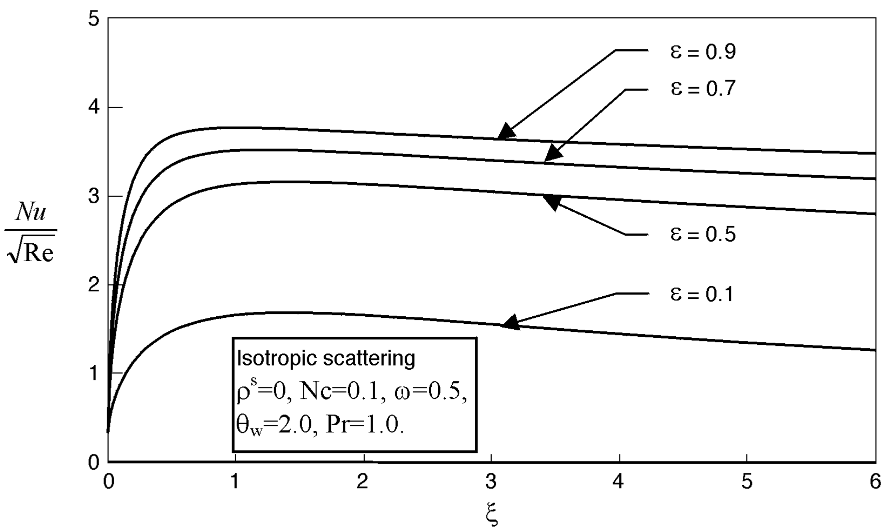

Figure 5.
Effect of emittance

Conclusion

Radiation interaction with forced laminar convection from a flat plate to an absorbing, emitting and anisotropically scattering gray medium with the surface of the plate reflecting radiation in diffuse-specular fashion is analyzed. The radiative transfer equation is solved with the discrete ordinates method. The partial differential system defining the governing dimensionless momentum and energy equations is solved by a finite difference method. Results are presented for the effects of degree of anisotropy in scattering,

scattering albedo, emittance, and the nature of surface reflection on the total heat flux from the plate to the fluid. The following conclusions are drawn from the analysis:

- Degree of radiation scattering anisotropy does not affect the total heat flux from the plate to the fluid significantly.
- Effect of nature of radiation reflection from the surface of the plate is negligible.

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